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The "magic" rule spaces of neural-like elementary cellular automata

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Abstract

A subclass of elementary cellular automata (CA), equivalent to a particular class of neural networks (NN) is studied. The NN are dependent on four real parameters a, b, c and τ; the subspace of the rule space which is obtained by varying the parameters and the dynamical characteristics of the rules belonging to this subset are determined. Finally, a few nice and "magic" regularities in this subset are noted and explained.

1. Introduction

Cellular automata (CA) are used in physics as mathematical models of complex systems because, despite their simple structure, they can simulate even very complicated phenomena. Also in theoretical computer science CA are employed as abstract models of parallel computational devices. In both kinds of applications it is essential to understand their behavior and thus to look at them as iterated dynamical systems.

In this approach, an important question is to give a classification of CA based on their dynamics. Wolfram [18] gave the first answer to this question by dividing all the one dimensional CA into four classes, after looking at extensive simulations. His approach is essentially empirical, the classes he introduced are not defined in a mathematical way; in order to assign a CA to a class it is necessary to look at the space–time patterns generated during the dynamical evolution starting from particular initial configurations. Furthermore, the choice of the class which each automaton belongs to is in some sense "subjective"; in fact, as the definition of the classes is not formal, there could be a certain "degree of freedom" in deciding what the class description means.

Since Wolfram introduced his classification, a lot of different approaches have been followed to give an answer to the same question [15, 11, 13] but till now, this is still an open problem.

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A question related to the problem of classification is the characterization of chaos in CA; it is possible to follow at least three approaches in looking for chaos in CA. According to the first one, introduced by Wolfram, one calls chaotic those CA that generate complex and aperiodic space–time patterns (CA belonging to the third class of Wolfram classification).

A second possible characterization of chaos for CA, introduced by Packard in [12], is the possession of a positive Lyapunov exponent (which is mathematically well defined, but whose values are calculated on the basis of a finite number of initial configurations, for a perturbed configuration belonging to a narrow neighbourhood of the chosen state, and for a finite number of iterations). These two approaches are empirical; the Wolfram's one is not founded on a mathematical definition of chaos, whereas Packard gives a numerical computation of Lyapunov exponents; but, both are based on finite number of empirical tests. Anyway, they seem to agree on what rules are claimed "chaotic".

The third way to chaos, the one we follow in this paper, is to look for CA satisfying a rigorous mathematical definition. We chose the Devaney definition of chaotic (deterministic) discrete time dynamical sytems [8], since it seems to have been accepted as standard by many scientists (for instance [16, 1]). This definition consists of three properties: *transitivity*, *regularity* and *sensitive dependence on initial conditions* (where the former two are topological conditions, and the latter is a metric one). We stress the fact that in the Devaney approach to chaos, the shift map is the prototypical example of chaotic dynamical system. Thus, all CA with a shift-like dynamics, which are not considered chaotic by Wolfram since they do not show aperiodic (complex) space–time pattern, exhibit a component of chaos according to the above-mentioned mathematical definition.

In this paper we introduce a tool that can be helpful in the study of CA as dynamical systems, in order to make an attempt to solve the problem of their classification. We introduce a class of elementary CA equivalent to a particular kind of neural networks (NN), called neural cellular automata (NCA). These NCA are dependent on four real parameters, a, b, c, and τ, and we study their qualitative dynamics varying the parameter values. Essentially two kinds of dynamics are observed: stable (characteristic of the CA whose dynamical evolution is attracted by fixed or periodic points) and chaotic, according to Devaney's definition (typical of subshift CA); it is interesting to note that none of these NCA generate the complex patterns called "chaotic" in Wolfram's approach. This lead us to suppose that the space–time pattern complexity is related to a computational complexity of the local rule, which cannot be computed by the simple NCA considered here.

We also observe the structure of the rule space obtained for each value of the parameters and note, and in part explain, some regularities that make this space "magic".

The paper is structured in the following way: in Section 2 we introduce the definition of cellular automaton and the particular class of NN that we shall consider and we show the equivalence of the NN introduced with a class of elementary CA. In Section 3 we study in detail the space of NCA obtained with a null threshold vector: in this section we present and prove results which can be extended in a straightforward way to NCA with nonnull threshold vector. The results for this latter general case are presented in Appendices A and B: we do not present any proof since it would be an unnecessary repetition of the reasoning followed in Section 3. In Section 4 we introduce another kind of tool to classify CA and apply them to NCA, comparing the results obtained in the different approaches. In Section 5 we draw some conclusions.

2. Basic definitions

We shall consider only bi-infinite, one-dimensional cellular automata, that is the sites are on a straight line.

Definition 1. A *bi-infinite, one-dimensional cellular automaton* or CA is a structure

$$\mathscr{C} = \langle \mathbf{Z}, G, r, h \rangle$$

where:
$\mathbf{Z} = \{\dots, -i, \dots, 0, \dots, i, \dots\}$ is the set of cells; $i \in \mathbf{Z}$ is the *location* of cell 'i';
$G = \{0, 1, \dots, k-1\}$ is the set of possible *states* of the cells;
$r \in \mathbf{N}$ is the *radius of the neighborhood*;
$h : G^{2r+1} \to G$ is the *local function*, also called the *rule* of the automaton.

A *configuration* of a CA is a mapping that specifies a state for each site of the lattice

$$\underline{x} : \mathbf{Z} \to G$$

and can be represented by a bi-infinite sequence:

$$\underline{x} = (\dots, x_{-m}, x_{-m+1}, \dots, x_{-1}, |x_0, x_{+1}, \dots, x_{m-1}, x_m, \dots).$$

On the configuration spaces $G^{\mathbf{Z}}$ we can define the *global function* of the CA:

$$\underline{g} : G^{\mathbf{Z}} \to G^{\mathbf{Z}}$$

which associates to any configuration $\underline{x} \in G^{\mathbf{Z}}$, the configuration at the next time step:

$$\underline{g}(\underline{x}) = (\dots, g_{i-1}(\underline{x}), g_i(\underline{x}), g_{i+1}(\underline{x}), \dots) \in G^{\mathbf{Z}}$$

The i-component $g_i : G^{\mathbf{Z}} \to G$ of the global function $\underline{g}$ specifies the next state of site i according to the following law, which involves the local rule h applied to the states of all the cells of the r-neighbourhood of the site i:

$$\forall i \in \mathbf{Z} \quad g_i(\underline{x}) := h(x_{i-r}, x_{i-r+1}, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{i+r-1}, x_{i+r}).$$

The pair $(G^{\mathbf{Z}}, \underline{g})$ can be considered a (deterministic) *discrete time dynamical system* (DTDS), once equipped the *configuration* (also *phase* or *state*) *space* $G^{\mathbf{Z}}$ with a metric

$d : G^{\mathbf{Z}} \times G^{\mathbf{Z}} \mapsto \mathbf{R}_+$ such that the *transition* (also *next state*) *map* $\underline{g} : G^{\mathbf{Z}} \to G^{\mathbf{Z}}$ turns out to be continuos (for the definition of the metric in the general case of a set G containing more than two values, see [16], where it is also shown that under the chosen metric the phase space $G^{\mathbf{Z}}$ is a Cantor set; we will introduce in the following the concrete metric for the particular boolean case $G = \{0, 1\}$ used in this paper). The positive motion of initial state $\underline{x} \in G^{\mathbf{Z}}$ is a mapping $\gamma_{\underline{x}} : \mathbf{N} \to G^{\mathbf{Z}}$, which tells us in which state the system is at each time t, if at time 0 it was in state $\underline{x}$:

$$\forall t \in \mathbf{N} \quad \gamma_{\underline{x}}(t) = \underline{g}^t(\underline{x})$$

Example 1. Let $G = \{0, \dots, k-1\}$ and let us consider the local rule of a CA defined by

$$h(x_{-r}, \dots, x_0, \dots, x_r) = x_1$$

The global transition function of this CA is the shift map $\sigma : G^{\mathbf{Z}} \to G^{\mathbf{Z}}$, that is $\forall i \in \mathbf{Z}$, $\sigma_i(\underline{x}) = x_{i+1}$. In a similar way we can introduce the local rule $h_R(x_{-r}, \dots, x_0, \dots, x_r) = x_{-1}$ whose corresponding global transition function is the right shift map σ_R: for each $i \in \mathbf{Z}$, $(\sigma_R)_i(\underline{x}) = x_{i-1}$.

Since we are going to study the qualitative dynamical behavior of these CA, we introduce some definitions that will be useful:

Definition 2. A subset A of the configuration space $G^{\mathbf{Z}}$ is *positively invariant* (resp., *strictly positively invariant*) iff $\underline{g}(A) \subseteq A$ (resp., $\underline{g}(A) = A$).

If A is positively invariant, then it is trivially true that $\forall t \in \mathbf{N}$, $\underline{g}^t(A) \subseteq A$; moreover, if A is positively invariant, then also its topological closure $\overline{A}$ is positively invariant. For any positively invariant subset A, the pair $(A, \underline{g})$ can be considered a dynamical system by its own, i.e., a dynamical subsystem of $(G^{\mathbf{Z}}, \underline{g})$, since any positive motion of initial state $\underline{a} \in A$ is such that $\forall t \in \mathbf{N}$, $\gamma_{\underline{a}}(t) = \underline{g}^t(\underline{a}) \in A$. In a certain sense a positively invariant set A is a "trapping" set since if a positive motion of initial state $\underline{x} \in G^{\mathbf{Z}}$ falls into A for the first time at instant $t_0 \in \mathbf{N}$ (i.e., $\gamma_{\underline{x}}(t_0) \in A$), then from now on the orbit is trapped to stay into A (i.e., $\forall t > t_0$, $\gamma_{\underline{a}}(t) = \underline{g}^t(a) \in A$).

Definition 3. A positively invariant set $A \subset G^{\mathbf{Z}}$ is an *asymptotic global attractor* for a CA rule h if

$$\forall \underline{x} \in G^{\mathbf{Z}}, \quad \lim_{t \to \infty} d_s(\underline{g}^t(\underline{x}), A) = 0$$

where $d_s(\underline{y}, A) = \inf_{\underline{z} \in A} d(z, y)$. A is a *k-step global attractor* if

$$\forall \underline{x} \in G^{\mathbf{Z}}, \quad \underline{g}^k(\underline{x}) \in A$$

A is a *finite step global attractor* if

$$\forall \underline{x} \in G^{\mathbf{Z}}, \quad \exists n \in \mathbf{N} : \qquad \underline{g}^n(\underline{x}) \in A$$

We shall restrict our attention to *elementary CA*, that is $G = \{0,1\}$, $r = 1$; we denote by Σ the configuration space $\{0,1\}^{\mathbf{Z}}$ and we consider Tychonoff metric on the space Σ, that is: $\forall \underline{x}, \underline{y} \in \Sigma$,

$$d(\underline{x}, \underline{y}) = \sum_{i=-\infty}^{\infty} \frac{1}{4^{|i|}} |x_i - y_i|$$

With respect to this metric the configuration space is a Cantor set and every global function induced by elementary CA is continuous.

The local function of an elementary CA is a mapping $h : \{0,1\}^3 \to \{0,1\}$, thus there are exactly 256 elementary rules, each of them represented by a lookup table:

x, y, z	$h(x, y, z)$
0,0,0	$h(0,0,0)$
0,0,1	$h(0,0,1)$
0,0,0	$h(0,1,0)$
0,1,1	$h(0,1,1)$
1,0,0	$h(1,0,0)$
1,0,1	$h(1,0,1)$
1,1,0	$h(1,1,0)$
1,1,1	$h(1,1,1)$

or by a boolean vector of length 8: $(h(1,1,1), \ldots, h(0,0,0))$; to each rule h we assign the integer number n_h whose binary representation is this vector. We call the set $\{0,1\}^8$ the *rule space*. The rule space can be divided in equivalence classes according to the following transformations:

Definition 4. Let $h : \{0,1\}^3 \to \{0,1\}$ be an elementary rule, we denote by h^* the *conjugate* rule of h, defined in the following way:

$$\forall (x_{-1}, x_0, x_1) \in \{0,1\}^3 \quad h^*(x_{-1}, x_0, x_1) = 1 - h(1 - x_{-1}, 1 - x_0, 1 - x_1)$$

and by h° the *reflected* rule of h, defined in the following way:

$$\forall (x_{-1}, x_0, x_1) \in \{0,1\}^3 \quad h^\circ(x_{-1}, x_0, x_1) = h(x_1, x_0, x_{-1})$$

Finally let $h^{\circ *} (= h^{*\circ})$ the *reflected conjugate* rule of h:

$$\forall (x_{-1}, x_0, x_1) \in \{0,1\}^3 \quad h^{\circ *}(x_{-1}, x_0, x_1) = 1 - h(1 - x_1, 1 - x_0, 1 - x_{-1})$$

To each rule h we can associate the set $\mathscr{C}(h) = \{h, h^*, h^\circ, h^{\circ *}\}$; we observe that in some cases the set contains less than four rules since some rules are invariant with respect to the introduced transformations. The collection of sets $\{\mathscr{C}(h) : h \in \{0,1\}^8\}$ is a partition of the rule space and thus it induces an equivalence relation on it; in fact it is easy to see that each rule h belongs to the set $\mathscr{C}(h)$ and that if $h' \notin \mathscr{C}(h)$, then $\mathscr{C}(h') \cap \mathscr{C}(h) = \emptyset$.

The transformations introduced above give rise to topological conjugacy between dynamical systems, as expressed in the following:

Proposition 1. *Let $h : \{0,1\}^3 \to \{0,1\}$ be an elementary rule and h^* its conjugate, the dynamical systems $(\{0,1\}^{\mathbf{Z}}, \underline{g})$ and $(\{0,1^{\mathbf{Z}}, \underline{g}^*)\}$ (where $\underline{g}^*$ is the global transition function induced by h^*) are topologically conjugate, that is the following diagram commutes:*

$$\begin{array}{ccc} \{0,1\}^{\mathbf{Z}} & \xrightarrow{\underline{g}} & \{0,1\}^{\mathbf{Z}} \\ \phi^* \downarrow & & \downarrow \phi^* \\ \{0,1\}^{\mathbf{Z}} & \xrightarrow{\underline{g}^*} & \{0,1\}^{\mathbf{Z}} \end{array}$$

where $\phi^* : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$ defined as: $\forall \underline{x} \in \{0,1\}^{\mathbf{Z}}$, $\forall i \in \mathbf{Z}$ $\phi_i(\underline{x}) = 1 - x_i$ is a homeomorphism.

An analogous result holds for h° and $h^{\circ *}$, defining the homeomorphisms $\phi^\circ : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$ and $\phi^{\circ *} : \{0,1\}^{\mathbf{Z}} \to \{0,1\}^{\mathbf{Z}}$, respectively: $\forall \underline{x} \in \{0,1\}^{\mathbf{Z}}$, $\forall i \in \mathbf{Z}$,

$$\phi_i^\circ(\underline{x}) = x_{-i} \quad \text{and} \quad \phi_i^{\circ *}(\underline{x}) = 1 - x_{-i}$$

We define a mapping $\mathbf{M} : \{0,1\}^8 \to \{0,1\}^8$ such that for each rule h, $\mathbf{M}(h)$ is the minimum number among the rules in the equivalence class $\mathscr{C}(h)$ induced by h. As far as we are interested in the qualitative dynamical behavior of a CA this transformation does not affect our study, owing to the above conjugacies. Indeed, according to [16] "The importance of [topological conjugacies] in the study of dynamical systems cannot be overestimated. [...] If we consider general properties of dynamical systems, [topological conjugacies] provide us with a way of classifying dynamical systems according to properties which remain unchanged after a transformation." We quote, for instance, the fact that cyclic points are preserved with the same period, stability and asymptotical stability of orbits are preserved, and as we will see later also the properties defining deterministic chaos in discrete time dynamical systems are invariant under topological conjugacy.

In this paper we shall study a subset of the rule space, that is the set of elementary CA which can be simulated by a particular neural network.

Definition 5. A bi-infinite neural network can be defined as a structure:

$$\mathscr{R} = \langle \mathbf{Z}, G, W, \underline{\tau}, \{f_i : i \in \mathbf{Z}\} \rangle$$

where:

$\mathbf{Z} = \{\ldots -i, \ldots, 0, \ldots, i, \ldots\}$ is the set of *neurons*;

$G = \{0, 1, \ldots, k-1\}$ is the set of *states* of the neurons;

$W = (w_{ij})_{i,j \in \mathbf{Z}}$, $w_{ij} \in \mathbf{R}$, is the bi-infinite *connection matrix*, satisfying the condition: $\forall \underline{x} = (x_i)_{i \in \mathbf{Z}} \in G^{\mathbf{Z}}$ and $\forall i \in \mathbf{Z}$, $\sum_j w_{ij} x_j$ is convergent;

$\underline{\tau} \in \mathbf{R}^{\mathbf{Z}}$ is the *threshold* vector;

$f_i : \mathbf{R} \to G$ is the activation function of neuron i;

This network has a bi-infinite number of neurons; its global transition function is the mapping $\underline{g} : G^{\mathbf{Z}} \to G^{\mathbf{Z}}$, whose component functions $g_i : G^{\mathbf{Z}} \to G$ are defined as follows:

$$\forall i \in \mathbf{Z} \quad g_i(\underline{x}) = f_i\left(\sum_j w_{ij}x_j - \tau_i\right)$$

We consider a particular class of NN where $G = \{0, 1\}$ (binary NN) whose connection matrix has the form

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \dots & b & c & 0 & 0 & 0 & 0 & \dots \\ \dots & a & b & c & 0 & 0 & 0 & \dots \\ \dots & 0 & a & b & c & 0 & 0 & \dots \\ \dots & 0 & 0 & a & b & c & 0 & \dots \\ \dots & 0 & 0 & 0 & a & b & c & \dots \\ \dots & 0 & 0 & 0 & 0 & a & b & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with a, b, c real constants, and the threshold vector is the constant vector $\underline{\tau} = (\dots, \tau, \tau, \tau, \dots) \in \mathbf{R}^{\mathbf{Z}}$.

Furthermore we assume that all the neurons have the same activation function, the Heavyside function $HS : \mathbf{R} \to \{0, 1\}$ defined as $\mathrm{HS}(x) = 1$ if $x \geqslant 0$, $\mathrm{HS}(x) = 0$ otherwise. Under these assumptions each neuron changes its state on the basis of the states of its two adjacent neurons and itself according to the component transition functions:

$$g_i(\underline{x}) := \mathrm{HS}\left(ax_{i-1} + bx_i + cx_{i+1} - \tau\right).$$

This neural network is *local* (since the next state of each neuron depends only on the neighboring neurons) and *homogeneous* (since all the neurons update their state on the basis of the same function). A similar kind of NN have been studied by Goles in [10], relating the stable behavior of cylindric NCA to the existence of a Lyapunov function.

For any fixed choice of parameters (a, b, c, τ), this net is equivalent to a one dimensional, bi-infinite, CA in which every site evolves according to the local rule whose table is the following:

x	y	z	$h(x, y, z)$
0	0	0	$\mathrm{HS}(-\tau)$
0	0	1	$\mathrm{HS}(c - \tau)$
0	1	0	$\mathrm{HS}(b - \tau)$
0	1	1	$\mathrm{HS}(b + c - \tau)$
1	0	0	$\mathrm{HS}(a - \tau)$
1	0	1	$\mathrm{HS}(a + c - \tau)$
1	1	0	$\mathrm{HS}(a + b - \tau)$
1	1	1	$\mathrm{HS}(a + b + c - \tau)$

Varying the parameters a, b, c, τ in $\mathbf{R}$, we obtain different NN that are equivalent to different elementary CA; we note that not all the elementary CA can be obtained in this way, since in order to simulate some rules (for example rule 90) we need a NN computing a *XOR*, and this is impossible with a NN of this type.

In the sequel we call *Neural-like CA* (NCA) the elementary automata that can be simulated by a NN of this type. Each NCA is equivalent to infinite NN, because the activation function HS causes the same behavior for an infinity of different values of the parameters, thus we shall single out some regions of the space of the parameters, corresponding to particular NCA. Furthermore, we shall call *NN-like rule space* the subset of the rule space consisting of those elementary CA equivalent to a NN of the kind introduced above.

3. Chaos and shift in cellular automata

For dynamical systems there is no universally accepted definition of chaos.

Several different quantities have been introduced to single out chaotic phenomena, such as entropy and Lyapunov exponent, and many different properties have been proposed to characterize chaoticity of dynamical systems.

In the context of cellular automata, the question of chaos was first faced by Wolfram. The classification introduced by Wolfram [18] has become the starting point of many approaches. This classification is based on the type of space–time patterns produced by the dynamical evolution of one dimensional CA, starting from random generated initial configurations. After many simulations Wolfram divided the one dimensional CA into four classes, whose description is informal:

W_1 evolution leads to a homogeneous state, after a finite transient;

W_2 evolution leads to a set of separated simple stable or periodic (space–time) structures;

W_3 evolution leads to aperiodic ("chaotic") space–time patterns;

W_4 evolution leads to complex localized structures, sometimes long-lived.

It seems that none of the elementary CA is in class W_4. There are many problems related to this classification. First, it is undecidable [7], that is we cannot establish in an algorithmic way to which class each automaton belongs to. The only way to assign an automaton to a class is to look at the space–time patterns it creates; the dynamical evolution is observed during a finite number of time steps, starting from many (but finite) randomly generated initial configurations. The decision depends on the particular set of (finite) configurations we have empirically obtained; if we follow the same method another time, we are not sure to obtain the same results. Furthermore, the choice of the class a rule belongs to is in some sense "subjective" since the definition of the classes is not formal and thus one must decide what the class description means.

In the simulations carried out to classify one dimensional CA, Wolfram recognized "chaotic" phenomena in those CA generating aperiodic space–time patterns. This characterization of chaos is not founded on a formal definition nor on a computable quantity that individuate objectively the set of chaotic CA.

Afterwards, Packard [12] gave an algorithm to obtain a numerical approximation of Lyapunov exponent for those CA for which it is well-defined. The empirical results obtained for a finite very narrow number of CA seem to agree with Wolfram's description of chaos: that is rules with positive Lyapunov exponent seem to be the ones in Wolfram class 3. Anyway, in defining as chaotic those CA with positive Lyapunov exponent, empirical results, also in this case obtained for a finite number of initial configurations and for a finite number of iterations, cannot guarantee that *all* CA in Wolfram class 3 are chaotic according to this definition.

A third possibility is to give a formal, rigorous definition of chaos based on topological properties of a CA as dynamical system and then to try to single out the class of those CA satisfying these properties, possibly by some theorems linking the (finite) values of the local rule to the chaotic behavior of the dynamics produced by the global transition rule on the (infinite) configuration space. In this paper we follow this approach and refer to the definition of chaotic discrete time dynamical system introduced by Devaney [8].

Definition 6. A discrete time dynamical system $< X, g >$, where X is the metric state (phase) space and $g : X \mapsto X$ the continuous transition (next state) function, is *chaotic* iff it is

(i) Transitive. For each pair of nonempty open subset $U, V \subseteq X$ there exists an integer $n \geqslant 0$ s.t. $g^n(V) \cap U \neq \emptyset$.

(ii) Regular. the set of periodic points is a dense subset of the state space X.

(iii) Dependent on initial conditions. There exists a constant $\varepsilon > 0$ (the *sensitivity constant*) s.t. for each $x \in X$ and for each $\delta > 0$ there exists $y \in X$ and $n \geqslant 0$ with $d(x, y) < \delta$ and $d(g^n(x), g^n(y)) \geqslant \varepsilon$.

In [2] it has been shown that, for dynamical systems with infinite state space, the former two conditions imply the third one. Sensitive dependence on initial conditions is the most evident feature of chaos; for instance in [9] it is claimed that: "Sensitive dependence on initial conditions is also expressed by saying that the system is *chaotic*". If a system is dependent on initial conditions and we are not able to measure with infinite precision the initial state (that is instead of the initial state p, we measure a perturbed one), we cannot predict the dynamical evolution, because "no matter how small a neighborhood of p we consider, there is at least one point in this neighborhood such that after a finite number of iterations, p and this point have separated by some fixed distance" [16].

Hence "If a map possesses sensitive dependence on initial conditions, then for all practical purposes, the dynamics of the map defy numerical computation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may bear no resemblance whatsoever with the real orbit." [8].

Some remarks about "transitivity". A sufficient condition (easier to handle than the direct definition) for transitivity is the existence of an orbit dense in the phase space. Transitivity is in a certain sense an irreducibility or indecomposability condition, since it implies that the dynamical system cannot be split into two disjoint nontrivial subsystems.

Let us stress that conditions of "transitivity" and "regularity" are both preserved under topological conjugacy (see [8]), as they are pure topological properties. However, as shown by a counterexample in [2], there exists a pair of topologicaly conjugate DTDS such that one of the two shows sensitivity to initial conditions and the second one is not sensitive (sensitivity is a metric property preserved only by isometrical conjugations); moreover, the former is an example of a DTDS which is sensitive, but is neither transitive nor regular.

To avoid confusion, in the sequel we shall call *complex* the CA which show a Wolfram "chaotic" aperiodic space–time pattern, *Lyapunov chaotic* CA with positive Lyapunov exponent according to Packard approach, and *topologically chaotic* (simply *chaotic*) the CA which satisfy rigorous Devaney definition.

The paradigmatic example of a chaotic dynamical system according to Definition 6 is the shift, introduced in Example 1. As already seen, the shift map can be induced by a radius 1 cellular automaton. This automaton thus is "chaotic" even if the space–time pattern generated during its evolution is not "complex" (aperiodic) in the sense of Wolfram class 3.

Other examples of chaotic dynamical systems are, under suitable constraints, subshifts, which are dynamical subsystems of the shift whose state space is a closed, strictly σ-invariant subset of $G^{\mathbf{Z}}$ and such that the restriction to this subset of its global transition function is a shift mapping. (Let us notice that in [3] it is assumed as a definition of one dimensional dynamical chaos just a property, the so-called "*turbulence*", which in Theorem 15, Chapter II, turns out to be topologically equivalent to a subshift). Note that if a subshift, as dynamical subsystem, is a chaotic attractor then according to [14, 9] it is a *strange attractor.*

Some of the rules belonging to Wolfram class 2 have a dynamics that can be related to subshifts.

Definition 7. Let n, m be two positive integers, a *CA* $< G, Z, r, h >$ is an (n, m)-subshift if the set G^Z contains a nonempty closed $\underline{g}$-invariant subset $\Sigma_{(m,n)}$ s.t $\forall \underline{x} \in \Sigma_{(n,m)}$ $\underline{g}^n(\underline{x}) = \sigma^m(\underline{x})$.

In this paper we shall only use $(1, 1)$-subshifts (*simple subshifts*), $(2, 1)$-subshifts (*alternating subshifts*) and $(2,2)$-subshifts (*double alternating subshifts*).

As explained in [6] (see also [16]), given an (n, m)-subshift, the maximal set $\Sigma_{(n,m)}$ on which the dynamics is "shift-like" can be determined by a boolean matrix. It is possible to prove the following:

Theorem 1. *Let* $< G, Z, r, h >$ *a simple subshift; the subsystem* $< \Sigma_{(1,1)}, \underline{g} >$ *is chaotic iff the matrix describing* $\Sigma_{(1,1)}$ *is irreducible.*

Let $< G, Z, r, h >$ be an (n, m)-subshift; if the matrix describing $\Sigma_{(n,m)}$ is irreducible, then the subsystem $< \Sigma_{(n,m)}, \underline{g} >$ is chaotic.

For the sake of simplicity, we study in detail the subset of the NN-like rule space obtained for $\tau = 0$, since the regularities that make the space "magic" are qualitatively the same as for $\tau \leqslant 0$, while the case $\tau > 0$ is less interesting (and anyway their main results are presented in Appendices A and B).

4. The NN-like rule space for $\tau = 0$

In this section we study the NCA corresponding to $\tau = 0$. Since we want to study the dynamical behavior of NCA, in view of Proposition 1, we can choose for each NCA local rule h its minimal $\mathbf{M}(h)$; this leads us to study a smaller number of rules. If we want to draw a graphical representation of the set of NCA, we have to fix the values of one parameter; we have chosen to fix the value of b and to represent the rules in the (a, c)-plane. We note that, for a fixed value of $b < 0$ the set of rules obtained does not depend on the precise value of b, thus we can suppose to have fixed a value of b and study the (a, c) plane. The same is true for $b > 0$, thus we study separately the three case $b < 0$, $b = 0$ and $b > 0$.

4.1. *The case $b < 0$*

In the case $b < 0$ we obtain the following rules:

1. $a > 0$ and $c > 0$

$c > -b$ and $a < -b$	rule: 10111011 (187)
$c > -b$ and $a > -b$	rule: 11111011 (251)
$-(a+b) < c < -b$ and $a < -b$	rule: 10110011 (179)
$-(a+b) < c < -b$ and $a > -b$	rule: 11110011 (243)
$-(a+b) < c$ and $a < -b$	rule: 00110011 (51)

2. $a > 0$ and $c < 0$

$c > -a$ and $a < -b$	rule: 00110001 (49)
$-a < c < -(a+b)$ and $a > -b$	rule: 01110001 (113)
$c > -(a+b)$ and $a > -b$	rule: 11110001 (241)
$c < -a$ and $a < -b$	rule: 00010001 (17)
$c < -a$ and $a > -b$	rule: 01010001 (81)

3. $a < 0$ and $c > 0$

$c < -a$ and $c < -b$	rule: 00000011 (3)
$c > -a$ and $c < -b$	rule: 00100011 (35)
$-b < c < -a$	rule: 00001011 (11)
$-a < c < -(a+b)$ and $c > -b$	rule: 00101011 (43)
$c > -(a+b)$ and $c > -b$	rule: 10101011 (171)

4. $a < 0$ and $c < 0$

rule: 00000001 (1)

We denote by NCA^- the set of these rules.

Starting from this set of rules and applying the transformation $\mathbf{M}$ defined above we obtain the following rules

rule $34 = \mathbf{M}(187) = 187^*$	rule $32 = \mathbf{M}(251) = 251^*$	rule $50 = \mathbf{M}(179) = 179^*$
rule $34 = \mathbf{M}(243) = 243^{\circ *}$	rule $51 = \mathbf{M}(51)$	rule $35 = \mathbf{M}(49) = 49^\circ$
rule $43 = \mathbf{M}(113) = 113^\circ$	rule $42 = \mathbf{M}(241) = 241^{\circ *}$	rule $3 = \mathbf{M}(17) = 17^\circ$
rule $11 = \mathbf{M}(81) = 81^\circ$	rule $3 = \mathbf{M}(3)$	rule $35 = \mathbf{M}(35)$
rule $11 = \mathbf{M}(11)$	rule $43 = \mathbf{M}(43)$	rule $42 = \mathbf{M}(171) = 171^*$
rule $1 = \mathbf{M}(1)$		

We want to analyze the dynamical behavior of the rules in the set NCA^-; in view of Proposition 1, we can study only the rules of $\mathbf{M}(NCA^-)$. The rule space for $b < 0$ has been studied in [4], here we summarize the results of that work.

In Table 1 we show the dynamical behaviors exhibited by the NCA rules in this set. It have been detected five kinds of dynamical behavior.

Taking into account for each rule the minimal topologically conjugate we obtain the situation represented in Fig. 1, whose meaning is the following:

- fixed or periodic point
- simple subshift
- alternating subshift
- double alternating subshift

The transformation $\mathbf{M}$ makes "magic" the rule space, in fact observing Fig. 1, we can note three nice regularities:

Table 1
The dynamics of the neural-like rules with $b < 0$, minimal rules.

Dynamics of neural CA: $b < 0$	Rule number
Attracting fixed point with a unique repelling isolated cycle	32
Asymptotically global attracting unique cycle of period 2	50
One step global attracting cycles of period 2	1
Isolated cycles of period 2	51
One step global attracting simple left subshift	34, 42
Simple and alternating subshifts	3, 35
Simple and double alternating subshifts	11, 43

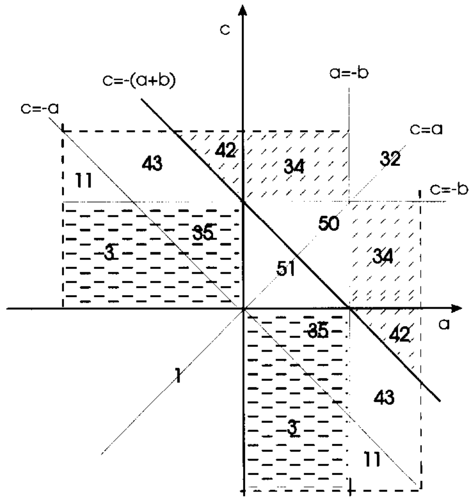

Fig. 1. The rule space for $b < 0$, minimal rules.

Fact 1. *The rule space* $\mathbf{M}(NCA^-)$ *is symmetric with respect to the line* $a = c$.

This is straightforward, since the symmetric of a point (a,c) with respect to this line is (c,a) and $h^\circ(x,y,z) = h(z,y,x) = \mathrm{HS}(az + by + cx)$, thus, in the original graphic, the rules over the line $a = c$ are those reflected from the corresponding ones under the line and they have the same minimal.

Fact 2. *In the space* $\mathbf{M}(NCA^-)$ *the rules over the line* $c = -(a+b)$ *are even, while the rules under this line are odd.*

If a rule h in NCA is over the line $c = -(a+b)$, it will be $h(1,1,1) = \mathrm{HS}(a+b+c) = 1$, thus the number corresponding to h will be greater than 128; furthermore, since $h(0,0,0) = 1$, $h^*(1,1,1) = h^{\circ *}(1,1,1) = 0$, thus the numbers corresponding to h^* and $h^{\circ *}$ are less than 128. We can conclude that $\mathbf{M}(h) \in \{h^*, h^{\circ *}\}$ and, since $h(1,1,1) = 1$, the number corresponding to $\mathbf{M}(h)$ is even. On the other hand, if h is a rule under the line $c = -(a+b)$, we have $h(1,1,1) = 0$, that is $h^*(0,0,0) = h^{\circ *}(0,0,0) = 1$ thus all the rules equivalent to h have an odd number, and in particular this is true for $\mathbf{M}(h)$.

Fact 3. *In the space* $\mathbf{M}(NCA^-)$ *a rule* h *immediately under the line* $c = -(a+b)$ *and the corresponding rule* h' *just over this line are such that* $n_h = n_{h'} + 1$.

Observing the rule space represented in Fig. 1 we can note an increasing complexity in the dynamics starting from a symmetric connection matrix lying in the region over the line $c = -(a+b)$ (even rules) and moving towards the antisymmetric ones, in the region of odd rules; in fact on the line $c = a$ we observe a simple dynamical behavior (fixed and periodic points), with a less simple behavior for rules 51 and 1 with respect to rules 50 and 32. Leaving the line $c = a$ we note the appearance of a shift-like dynamics, simple for even rules and "complex" for odd rules. Thus, starting from an even rule on the line $c = a$ we can move towards an odd rule obtaining a similar, but slightly more complex, dynamical behavior or towards an antisymmetric matrix observing the presence of a chaotic dynamics.

4.2. *The case $b = 0$*

Letting $b = 0$ we obtain the rules:

1. $a > 0$ and $c > 0$

   rule: 11111111 (255)

2. $a > 0$ and $c < 0$

   $c > -a$ rule: 11110101 (245)
   $c < -a$ rule: 00010101 (21)

3. $a < 0$ and $c > 0$

   $c > -a$ rule: 10101111 (175)
   $c < -a$ rule: 00090111 (7)

4. $a < 0$ and $c < 0$

   rule: 00000101 (5)

We denote by NCA^0 this set of rules. The minimal rules obtained by applying the transformation $\mathbf{M}$ are:

rule $0 = \mathbf{M}(255) = 255^*$	rule $10 = \mathbf{M}(245) = 245^{\circ *}$	rule $7 = \mathbf{M}(21) = 21^\circ$
rule $10 = \mathbf{M}(175) = 175^*$	rule $7 = \mathbf{M}(7)$	rule $5 = \mathbf{M}(5)$

The analysis of the dynamical behavior of these rules is in Table 2.
In Fig. 2 we give a graphical representation of the set NCA^-.
Also for $b = 0$ we have:

Fact 4. *The rule space $\mathbf{M}(NCA^0)$ is symmetric with respect to the line $a = c$.*

Table 2
The dynamics of the neural-like rules with $b = 0$, minimal rules

Dynamics of neural CA: $b = 0$	Rule number
One step global attracting unique fixed point	0
One step global attracting cycles	5
One step global attracting simple left subshift	10
Double alternating right subshift	7

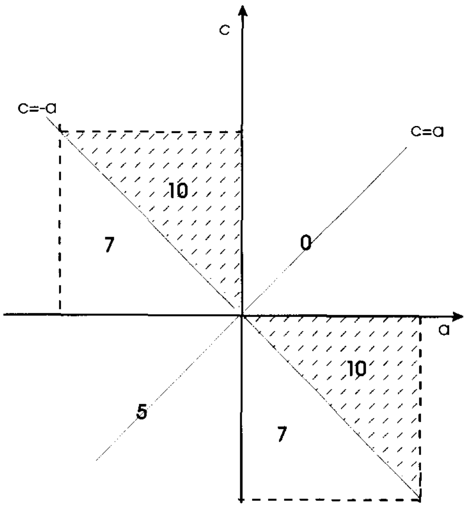

Fig. 2. The rule space for $b = 0$, minimal rules.

Fact 5. *In the space* $\mathbf{M}(NCA^0)$ *the rules over the line* $c = -a$ *are even, while all the rules under this line are odd.*

4.3. The case $b > 0$

The rules that we can obtain letting $b > 0$ are:

1. $a > 0$ and $c > 0$

   rule 11111111(255)

2. $a > 0$ and $c < 0$

$c > -a$ and $c > -b$	rule: 11111101 (253)
$c > -a$ and $c < -b$	rule: 11110101 (245)
$c < -a$ and $c > -b$	rule: 11011101 (221)
$c < -a$ and $c > -(a+b)$	rule: 11010101 (213)
$c < -(a+b)$	rule: 01010101 (85)

3. $a < 0$ and $c > 0$

$c > -a$ and $c > -b$	rule: 11101111 (239)
$c > -a$ and $c < -b$	rule: 10101111 (175)
$c < -a$ and $c > -b$	rule: 11001111 (207)
$c < -a$ and $c > -(a+b)$	rule: 10001111 (143)
$c < -(a+b)$	rule: 00001111 (15)

4. $a < 0$ and $c < 0$

$c > -(a+b)$	rule: 11001011 (205)
$c < -(a+b)$ and $c > -b$ and $a > -b$	rule: 01001101 (77)

$a > -b$ and $c < -b$ rule: 01000101 (69)
$a < -b$ and $c > -b$ rule: 00001101 (13)
$a < -b$ and $c < -b$ rule: 00000101 (5)

This set will be called NCA^+.
The application of the transformation $\mathbf{M}$ gives rise to the following set of rules:

rule $0 = \mathbf{M}(255) = 255^*$ rule $8 = \mathbf{M}(253) = 253^{\circ *}$ rule $10 = \mathbf{M}(245) = 245^{\circ *}$
rule $12 = \mathbf{M}(221) = 221^{\circ *}$ rule $14 = \mathbf{M}(213) = 213^{\circ *}$ rule $15 = \mathbf{M}(85) = 85^{\circ}$
rule $8 = \mathbf{M}(239) = 239^*$ rule $10 = \mathbf{M}(175) = 175^*$ rule $12 = \mathbf{M}(207) = 207^*$
rule $14 = \mathbf{M}(143) = 143^*$ rule $15 = \mathbf{M}(15)$ rule $76 = \mathbf{M}(205) = 205^*$
rule $77 = \mathbf{M}(77)$ rule $13 = \mathbf{M}(69) = 69^{\circ}$ rule $13 = \mathbf{M}(13)$
rule $5 = \mathbf{M}(5)$

The dynamical behavior of these rules is described in Table 3.
Fig. 3 show a pictorial representation of this set.
From Fig. 3 we can also see that:

Fact 6. *The rule space* $\mathbf{M}(NCA^+)$ *is symmetric with respect to the line* $a = c$.

Table 3
The dynamics of the neural-like rules with $b > 0$, minimal rules

Dynamics of neural CA: $b > 0$	Rule number
One step global attracting unique fixed point	0, 8
One step global attracting fixed points	12, 76
One step global attracting cycles	5
Global attracting periodic points	77
One step global attracting simple left subshift	10
Simple left and double alternating right subshift	14
Double alternating right subshift	15
Double alternating right subshift with asymptotically global attracting cycles of period 2	13

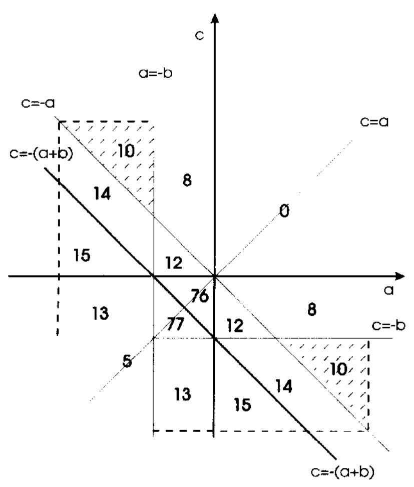

Fig. 3. The rule space for $b > 0$, minimal rules.

Fact 7. *In the space* $\mathbf{M}(NCA^+)$ *the rules over the line* $c = -(a+b)$ *are even, while the rules under this line are odd.*

Fact 8. *In the space* $\mathbf{M}(NCA^+)$ *a rule* h *just under the line* $c = -(a+b)$ *and the corresponding rule* h' *just over this line are such that* $n_h = n_{h'} + 1$.

If we observe how the set NCA changes varying the parameter b, we note that at the point $b = 0$ we have a bifurcation; in fact, starting from $b < 0$ and studying the dynamical behavior of the rules in the set NCA, we can see that in $b = 0$ the alternating subshift behavior disappears, thus varying the parameter b, we have the appearance of a new dynamical behavior for $b < 0$. In Figs. 4–6 we show how the rule space changes

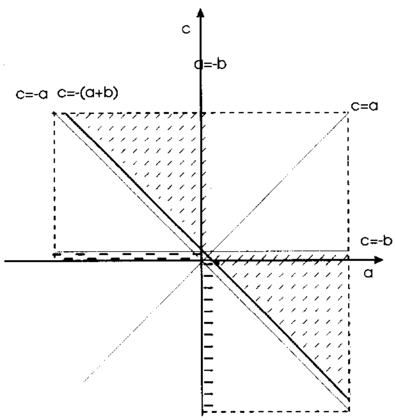

Fig. 4. The rule space for $b < 0$, b close to 0.

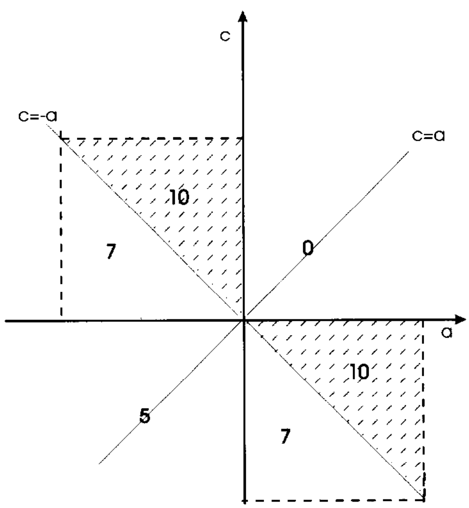

Fig. 5. The rule space for $b = 0$.

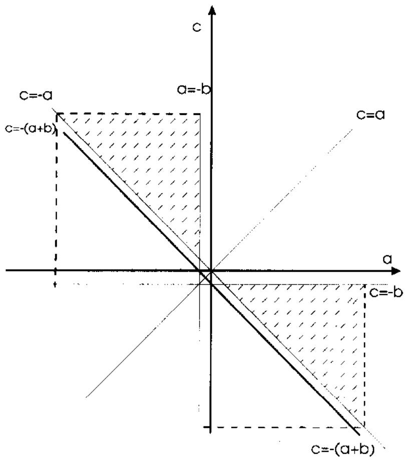

Fig. 6. The rule space for $b > 0$, b close to 0.

upon varying b, assuming values negative (4) and positive (6) very close to 0 and comparing the spaces obtained with the one for $b = 0$. In the appendices we show all the cases for $\tau \in \mathbf{R}$; one can see that the facts about the graphics for $\tau = 0$ are still true for $\tau \leqslant 0$, while for $\tau > 0$ the following properties hold:

Fact 9. *All the graphics corresponding to $\tau > 0$ are symmetric with respect to the line $a = c$.*

The proof is the same as in the case $\tau = 0$, since this fact does not depend on the value of τ.

Fact 10. *All the rules corresponding to $\tau > 0$ have an even number.*

If a rule h corresponds to $\tau > 0$, we have $h(0,0,0) = 0$, thus its conjugate h^* has a number greater than or equal to 128. If $h(1,1,1) = 0$, we have $\mathbf{M}(h) \in \{h, h^\circ\}$ and $\mathbf{M}(h)$ has an even number; if $h(1,1,1) = 0$ all the rules in $\mathscr{C}(h)$ have an even number, and in particular this holds for $\mathbf{M}(h)$.

5. A compared analysis

In this section we compare different approaches that have been used for the purpose of classification of elementary CA.

5.1. A classification of quiescent rules

In [5] we have introduced a classification of quiescent CA (i.e. CA for which $h(0,0,\ldots,0,0) = 0$) based on the pattern growth, that is on the way the length ℓ

of a finite configuration embedded in a null background

$$\underline{x} = (\dots, 0, 0, 1, \dots, 1, 0, 0, \dots)$$

is modified by the dynamical evolution of the CA. One dimensional quiescent CA are divided into three classes:

$\mathscr{C}_1$ is the set of rules h such that for each finite configuration $\underline{x}$ $\lim_{t\to\infty} \ell(\underline{g}^t(\underline{x})) = 0$.

$\mathscr{C}_2$ is the set of rules h such that for each finite configuration $\underline{x}$ $\sup_{t\in\mathbf{N}} \ell(\underline{g}^t(\underline{x})) < +\infty$.

$\mathscr{C}_2$ is the set of rules h such that for at least one finite configuration $\underline{x}$ $\sup_{t\in\mathbf{N}} \ell(\underline{g}^t(\underline{x})) = +\infty$.

This is a formal definition: the choice of the class is objective provided that we can know the behavior of the automaton with respect to the pattern growth.

Furthermore, limiting the study to elementary cellular automata, in [5] we have made this classification effective, that is we have given an algorithm to assign a quiescent rule to the class it belongs to; this is made relating the dynamical characteristics of the pattern growth to some static properties of the local rule.

In Table 4 we show all the NCA, divided on the basis of their qualitative dynamics, where in the case of quiescent NCA it is possible to compare the result of empirical Wolfram classification with the formal classification based on their pattern growth.

Two facts are evident from this table: none of the NCA is in class W_3 and only two of them are in class $\mathscr{C}_3$. This lead us to suppose that NCA are in some sense "simpler" than other CA; in fact, even if some of them are chaotic dynamical systems, we can easily describe their qualitative dynamics. The fact that most of them are in class $\mathscr{C}_1$ and $\mathscr{C}_2$ means that, at least starting from finite configurations, they reach their attractor in finite time steps; moreover we are able to say what will happen in this attractor. On the contrary, for rules in $\mathscr{C}_3$ the attractor is asymptotic while for rules in W_3 we are not able to describe the dynamical behavior in terms of periodic points or chaotic orbits: NCA, with only two exceptions (rules 50 and 178), are in this sense easier to understand.

6. Conclusions

This paper was inspired by [4]; we have investigated in details the rule space of NCA and we have discovered and explained their regularities. Moreover, we have extended the results of that work to the case $\tau \neq 0$. We claim that the main interesting fact in this study is that the elementary NCA do not generate complex patterns. In fact, the characteristics of rules in classes 1 and 2 of Wolfram's classification have been well analyzed [4, 5] in many different approaches, while it has not been found to be an useful tool to understand the dynamical behavior of rules in class W_3: we conjecture that the complexity of the space–time patterns generated by the evolution of the rules from W_3 (of course, according to the definition of pattern complexity given in the Introduction, and different from topological chaoticity of the dynamical behavior)

Table 4

NCA	Dynamical behavior on bi-infinite configurations	Wolfram's class	Pattern growth class
0	The null configuration is a one step global attractor	W_1	$\mathscr{C}_1$
8		W_1	$\mathscr{C}_1$
4	The set of fixed points is a finite steps global attractor	W_2	$\mathscr{C}_2$
12		W_2	$\mathscr{C}_2$
36		W_2	$\mathscr{C}_2$
76		W_2	$\mathscr{C}_2$
200		W_2	$\mathscr{C}_2$
128	The set of fixed points is an asymptotic global attractor	W_1	$\mathscr{C}_1$
136		W_1	$\mathscr{C}_1$
140		W_2	$\mathscr{C}_2$
1	The set of periodic points is a finite steps global attractor	W_2	—
5		W_2	—
32		W_1	$\mathscr{C}_1$
51		W_2	—
13	The set of periodic points is an asymptotic global attractor	W_2	—
19		W_2	—
23		W_2	—
50		W_2	$\mathscr{C}_3$
77		W_2	—
160		W_1	$\mathscr{C}_1$
168		W_1	$\mathscr{C}_1$
178		W_2	$\mathscr{C}_3$
232		W_2	$\mathscr{C}_2$
2	Simple subshift Σ_0 finite step global attractor	W_2	$\mathscr{C}_2$
10		W_2	$\mathscr{C}_2$
34		W_2	$\mathscr{C}_2$
42		W_2	$\mathscr{C}_2$
138		W_2	$\mathscr{C}_2$
170		W_2	$\mathscr{C}_2$
162	Σ_0 asymptotic global attractor	W_2	$\mathscr{C}_2$
	Alternating subshift:		
3	Σ_1 global attractor	W_2	—
35	Σ_1 not attractor	W_2	—
	Double alternating subshift:		
15	Σ_2 global attractor	W_2	—
7	Σ_2 not attractor	W_2	—
14		W_2	$\mathscr{C}_2$
43		W_2	—
142		W_2	$\mathscr{C}_2$
11	Alternating and double alternating subshift	W_2	—

is related to a computational complexity of the local rule. We think that this can be a first step in order to single out the peculiarity of this class.

We are now looking for a tool that can help in the analysis of the rules that generate complex patterns.

Appendix A. The rule space for $\tau \leqslant 0$

We present the different cases in Tables 5–7 in particular for each case we draw a table which represents the minimal rules obtained applying the function **M** and a description of their dynamical behavior. In each table we only analyze the rules that do not appear in cases already analyzed; thus for the cases that give only rules already studied we do not draw any table. For each case we also draw a graphic representation of the rule space (Figs. 7–10).

Table 5
The case $\tau \leqslant b \leqslant 0$: minimal rules

Rule number	Dynamics
0	The null configuration is one step global attracting fixed point
1,5	The set of cycles of order 2 is a one step global attractor
23	The set of periodic points of order 2 is a global attractor
2, 10	The set Σ_0 is a one step global attractor
3	The closed invariant set Σ_1 contains Σ_0 and is a one step global attractor
7	The phase space contains a closed invariant subset Σ_2
11	The phase space can be divided into four subsets: Σ_0, Σ_1 Σ_2 and $\tilde{\Sigma} = \Sigma \setminus (\Sigma_0 \cup \Sigma_1 \cup \Sigma_2)$
15	On the whole phase space $\underline{g}^2 = \sigma^2$

Table 6
The rule space for $b < \tau \leqslant 0$: minimal rules

Rule number	Dynamics
19	The set of order at most 2 periodic points is an asymptotic global attractor
32	The points $(\ldots,0,1\vert 0,1,0,\ldots)$ and $(\ldots 1,0\vert 1,0,1,\ldots)$ are a cycle of order 2, while the null configuration $\underline{0} = (\ldots,0,0,0,\ldots)$ is an attractor for all the other configurations
50	The cycle consisting of the points $(\ldots,0,1\vert 0,1,0,\ldots)$ and $(\ldots 1,0\vert 1,0,1,\ldots)$ is a global attractor
51	$\forall x,y,z \in \{0,1\}^3$, $g(x,y,z) = 1-y$, the whole phase space is made of isolated cycles of order 2
34,42	The closed invariant set $\Sigma_0 = \{\underline{x} \in \Sigma : \underline{g}(\underline{x}) = \sigma(\underline{x})\}$ is a one step global attractor
35	The phase space can be divided in three subsets: Σ_0, Σ_1 and $\tilde{\Sigma} = \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$; there is at least one configuration that never goes in Σ_0 neither in Σ_1
43	The phase space can be divided in three subsets: Σ_0, Σ_2 and the set $\tilde{\Sigma} = \Sigma \setminus (\Sigma_0 \cup \Sigma_2)$

Table 7
The case $\tau \leqslant 0 \leqslant b$ and $\vert\tau\vert < b$: minimal rules

Rule number	Dynamics
8	The null configuration is a one step global attractor
14	The phase space contains a closed invariant subset Σ_2
4,12	The set of fixed points is a one step global attractor
13	The phase space contains a non empty set Σ_2 and the set of order at most 2 periodic points is an asymptotic global attractor
77	The set of order at most 2 periodic points is a global attractor

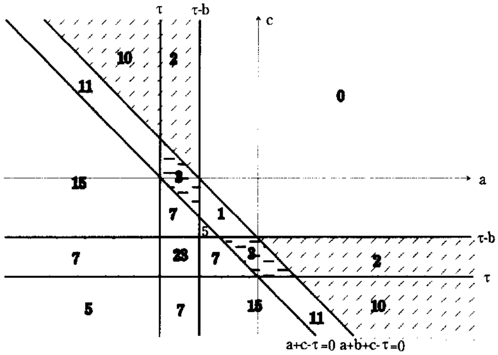

Fig. 7. The Rule Space for $\tau \leqslant b \leqslant 0$.

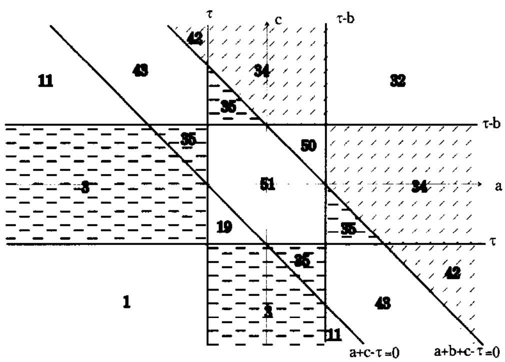

Fig. 8. The rule space for $b < \tau \leqslant 0$.

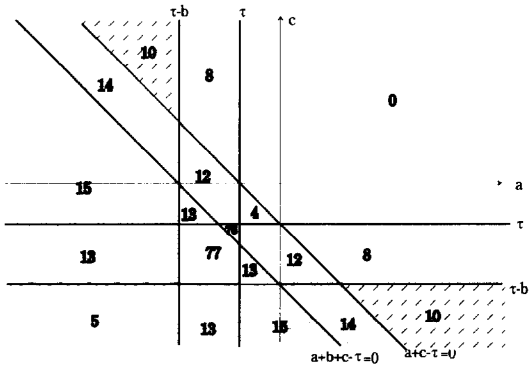

Fig. 9. The rule space for $\tau \leqslant 0 \leqslant b$ and $|\tau| < b$.

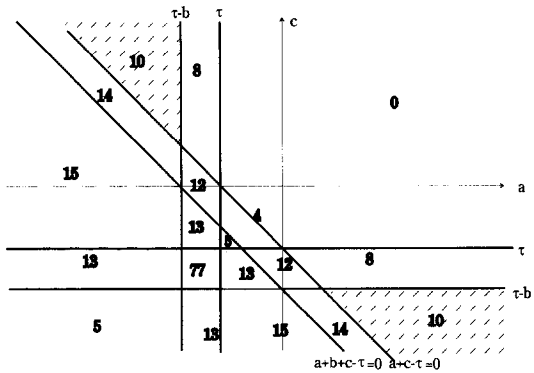

Fig. 10. The rule space for $\tau \leqslant 0 \leqslant b$ and $|\tau| \geqslant b$.

Appendix B: The rule space for $\tau > 0$

In this appendix we show the NCA corresponding to $\tau > 0$. The results are given in Tables 8–10 and Figs. 11–15.

Table 8
The case $0 < \tau \leqslant b$: minimal rules

Rule numbers	Dynamics
128	The configuration $(\dots 1,1,1,\dots)$ is an isolated fixed point; all the other points are attracted by the null configuration
36	The set of fixed points is a 2 steps global attractor
140	The set of fixed points is an asymptotic global attractor
138	The set Σ_0 is a one step global attractor
142	The phase space contains a closed invariant set Σ_2
76,200	The set of fixed points is a one step global attractor
204	The global map $\underline{g}$ is the identity, thus all the phase space is made of fixed points

Table 9
The case $0 < \frac{\tau}{2} \leqslant b < \tau$: minimal rules

Rule number	Dynamics
160,168	The set of order at most 2 periodic points, consisting of the null configuration, and the points $(\dots,1,1,1,\dots)$, $(\dots,1,0,1|0,1,\dots)$ and $(\dots,1,0,1,0|1,0,1,\dots)$, is a global attractor
170	On the whole phase space $\underline{g} = \sigma$
232	The points $(\dots,1,0,1,0|,1,\bar{0},\dots)$ and $(\dots,0,1,0,1|,0,1,\dots)$ are no isolated order 2 cycle, while all the other configurations are attracted by the set of fixed points
136	The set of fixed points is a global attractor

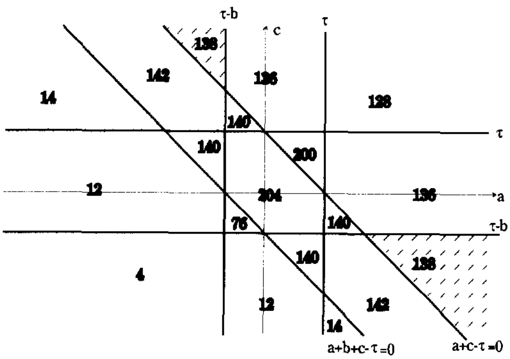

Fig. 11. The rule space for $0 < \tau \leqslant 0 \leqslant b$.

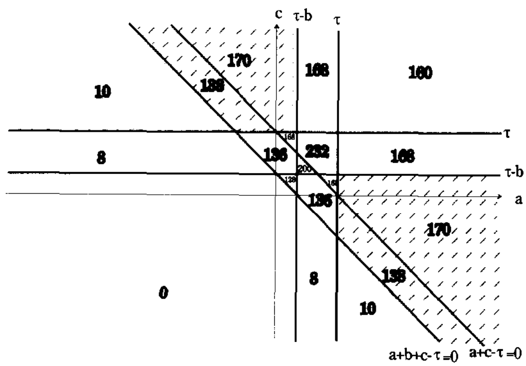

Fig. 12. The rule space for $0 < \frac{\tau}{2} \leqslant b < \tau$.

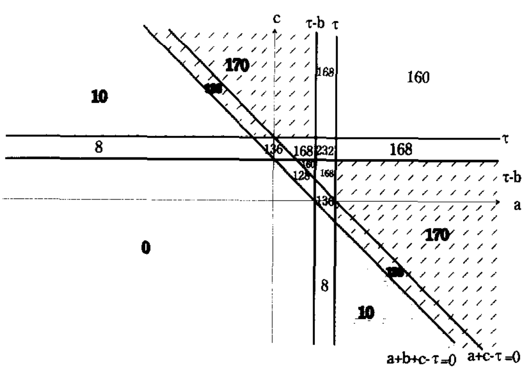

Fig. 13. The rule space for $0 < b < \frac{\tau}{2}$.

Table 10
The case $b \leqslant 0 < \tau$ and $|b| \geqslant \tau$: minimal rules

Rule number	Dynamics
162	The set Σ_0 is an asymptotic global attractor
178	The set of at most order 2 periodic points is an asymptotic global attractor

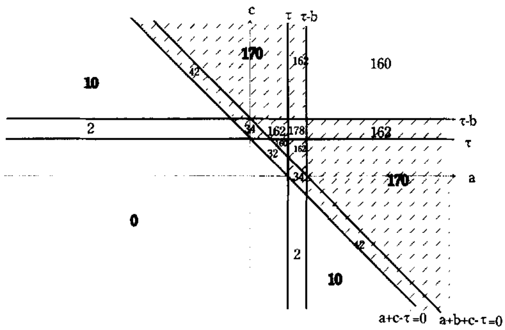

Fig. 14. The rule space for $b \leqslant 0 < \tau$ and $|b| \geqslant \tau$.

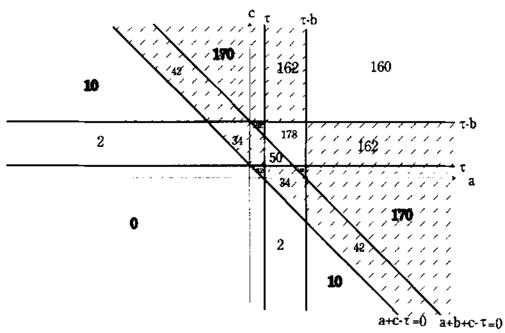

Fig. 15. The rule space for $b \leqslant 0 < \tau$ and $|b| < \tau$.

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